national accelerator laboratory

FN-221
0100

MOMENT EQUATIONS OF A DISTRIBUTION OF PARTICLES PERFORMING HARMONIC OSCILLATIONS

L. C. Teng

January 11, 1971

The moment equations have been studied by F. J. Sacherer (CERN/SI/Int.DL/70-12, 18.11.1970) up to the second moment. Here we extend it to higher moments.

Let the one-dimensional no space-charge single-particle equations be

$$\begin{cases} x' = p \\ p' = -kx \end{cases} \qquad \text{(prime means } \tfrac{d}{dt}\text{)} \tag{1}$$

and the distribution function be $\psi(x,p,t)$ which obeys the continuity equation

$$\frac{\partial\psi}{\partial t} + \frac{\partial}{\partial x}(x'\psi) + \frac{\partial}{\partial p}(p'\psi) = 0 \tag{2}$$

For now we shall consider k to be time independent.

MOMENT EQUATIONS

The equations for the various moments of the distribution can be derived simply as follows:

First Moment

$$\begin{cases} (\bar{x})' = \dfrac{d}{dt}\displaystyle\int x\ \psi(x,p,t)\ dx\ dp \\ \qquad = \displaystyle\int x\ \frac{\partial\psi}{\partial t}\ dx\ dp \\ \qquad = -\displaystyle\int x\left[\frac{\partial}{\partial x}(x'\psi) + \frac{\partial}{\partial p}(p'\psi)\right] dx\ dp \end{cases} \tag{3}$$

Operated by Universities Research Association Inc. Under Contract with the United States Atomic Energy Commission

$$\left| \begin{array}{l} \qquad = \int x'\psi \; dx \; dp = \bar{x}' = \bar{p} \\ (\bar{p})' = \bar{p}' = -k\bar{x} \end{array} \right.$$

where we have performed a partial integration and made use of the assumption that $\psi \to 0$ as $|x| \to \infty$ or $|p| \to \infty$. We can write these two equations as

$$X_1' = K_1 \, X_1 \tag{4}$$

where

$$X_1 \equiv \begin{pmatrix} \bar{x} \\ \bar{p} \end{pmatrix} \qquad K_1 \equiv \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}$$

Second Moment

$$\begin{cases} (\overline{x^2})' = 2 \, \overline{xx'} = 2 \, \overline{xp} \\ (\overline{xp})' = \overline{xp'} + \overline{x'p} = -k \, \overline{x^2} + \overline{p^2} \\ (\overline{p^2})' = 2 \, \overline{pp'} = -2k \, \overline{xp} \end{cases} \tag{5}$$

Or we can write

$$X_2' = K_2 \, X_2 \tag{6}$$

where

$$X_2 \equiv \begin{pmatrix} \overline{x^2} \\ \overline{xp} \\ \overline{p^2} \end{pmatrix} \qquad K_2 \equiv \begin{pmatrix} 0 & 2 & 0 \\ -k & 0 & 1 \\ 0 & -2k & 0 \end{pmatrix}$$

Similarly, we get the equations up to, say, the fifth moment. They are summarized as

$$X_n' = K_n \, X_n \tag{7}$$

where

$$X_1 = \begin{pmatrix} \overline{x} \\ \overline{p} \end{pmatrix} \qquad K_1 = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} \overline{x^2} \\ \overline{xp} \\ \overline{p^2} \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 2 & 0 \\ -k & 0 & 1 \\ 0 & -2k & 0 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} \overline{x^3} \\ \overline{x^2p} \\ \overline{xp^2} \\ \overline{p^3} \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 3 & 0 & 0 \\ -k & 0 & 2 & 0 \\ 0 & -2k & 0 & 1 \\ 0 & 0 & -3k & 0 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} \overline{x^4} \\ \overline{x^3p} \\ \overline{x^2p^2} \\ \overline{xp^3} \\ \overline{p^4} \end{pmatrix} \qquad K_4 = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ -k & 0 & 3 & 0 & 0 \\ 0 & -2k & 0 & 2 & 0 \\ 0 & 0 & -3k & 0 & 1 \\ 0 & 0 & 0 & -4k & 0 \end{pmatrix}$$

$$X_5 = \begin{pmatrix} \overline{x^5} \\ \overline{x^4p} \\ \overline{x^3p^2} \\ \overline{x^2p^3} \\ \overline{xp^4} \\ \overline{p^5} \end{pmatrix} \qquad K_5 = \begin{pmatrix} 0 & 5 & 0 & 0 & 0 & 0 \\ -k & 0 & 4 & 0 & 0 & 0 \\ 0 & -2k & 0 & 3 & 0 & 0 \\ 0 & 0 & -3k & 0 & 2 & 0 \\ 0 & 0 & 0 & -4k & 0 & 1 \\ 0 & 0 & 0 & 0 & -5k & 0 \end{pmatrix}$$

DIAGONAL FORMS OF MOMENT EQUATIONS

Either by direct computation or by decomposing K_n into operators similar to the creation and annihilation operators for Bosons and taking advantage of simple relationships between these operators, we get ($\omega^2 \equiv k$)

$$
\begin{cases}
K_1{}^2 + \omega^2 = 0 \\
K_2(K_2{}^2+4\omega^2) = 0 \\
(K_3{}^2+\omega^2)\ (K_3{}^2+9\omega^2) = 0 \\
K_4(K_4{}^2+4\omega^2)\ (K_4{}^2+16\omega^2) = 0 \\
(K_5{}^2+\omega^2)\ (K_5{}^2+9\omega^2)\ (K_5{}^2+25\omega^2) = 0
\end{cases}
\tag{8}
$$

The regularities of these equations are obvious. These relations show that the moments satisfy the following non-matrix (diagonal) linear equations

$$
\begin{cases}
X_1'' + \omega^2 X_1 = 0 \\
(X_2''+4\omega^2 X_2)' = X_2''' + 4\omega^2 X_2' = 0 \\
(X_3''+9\omega^2 X_3)'' + \omega^2(X_3''+9\omega^2 X_3) \\
\qquad = X_3'''' + 10\omega^2 X_3'' + 9\omega^4 X_3 = 0 \\
\left[(X_4''+16\omega^2 X_4)'' + 4\omega^2(X_4''+16\omega^2 X_4)\right]' \\
\qquad = X_4^{V} + 20\omega^2 X_4''' + 64\omega^4 X_4' = 0 \\
\left[(X_5''+25\omega^2 X_5)'' + 9\omega^2(X_5''+25\omega^2 X_5)\right]'' \\
\qquad + \omega^2\left[(X_5''+25\omega^2 X_5)'' + 9\omega^2(X_5''+25\omega^2 X_5)\right]'' \\
\qquad = X_5^{VI} + 35\omega^2 X_5'''' + 259\omega^4 X_5'' + 225\omega^6 X_5 = 0
\end{cases}
\tag{9}
$$

The equations for the lower moments are familiar. For example, X_1 equation gives

$$(\bar{x})'' + \omega^2\bar{x} = 0 \tag{10}$$

which simply states that the center of gravity of the distribution oscillates as a single particle. The X_2 equation gives

$$(\overline{x^2})'' + 4\omega^2(\overline{x^2})' = 0 \tag{11}$$

which is reminiscent of the equation for the Courant-Snyder β function.

BILINEAR INVARIANTS

We can define the bilinear invariants for the nth moment by

$$I_n \equiv \frac{1}{2}\,\tilde{X}_n S_n X_n \tag{12}$$

where $\tilde{}$ means transposition. The condition for invariance gives

$$\begin{aligned} 2\,I_n' &= \tilde{X}_n' S_n X_n + \tilde{X}_n S_n X_n' \\ &= \tilde{X}_n \tilde{K}_n S_n X_n + \tilde{X}_n S_n K_n X_n = 0 \end{aligned} \tag{13}$$

or

$$\tilde{K}_n S_n + S_n K_n = 0 \tag{14}$$

It can be shown directly that S_n has the following forms

$$\left\{ S_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right.$$

$$
\left\{
\begin{aligned}
S_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
S_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
S_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
S_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & -10 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\right. \tag{15}
$$

The regularity is, again, obvious. They give the invariants

$$
\left\{
\begin{aligned}
I_1 &= 0 \\
I_2 &= \overline{x^2}\ \overline{p^2} - \left(\overline{xp}\right)^2 \\
I_3 &= 0 \\
I_4 &= \overline{x^4}\ \overline{p^4} - 4\ \overline{x^3p}\ \overline{xp^3} + 3\left(\overline{x^2p^2}\right)^2 \\
I_5 &= 0
\end{aligned}
\right. \tag{16}
$$

The invariant I_2 can be defined as a measure of the mean-squared emittance.

ENVELOPE EQUATIONS

We can define the nth envelope ξ_n (envelope of the nth moment) by

$$\xi_n \equiv \left(\overline{x^n}\right)^{\frac{1}{n}} \tag{17}$$

and derive second order equations for these envelopes. (Strictly speaking, ξ_n can be interpreted as an "envelope" only for even values of n.) We shall demonstrate the procedure only for ξ_2

$$\begin{aligned}
\left(\overline{x^2}\right)'' &= 2\left(\overline{xp}\right)' = -2\,\omega^2\,\overline{x^2} + 2\,\overline{p^2} \\
&= -2\,\omega^2\,\xi_2^2 + 2\,\overline{p^2} \\
&= -2\,\omega^2\,\xi_2^2 + \frac{2}{\xi_2^2}\,\overline{x^2}\;\overline{p^2}
\end{aligned}$$

$$\begin{aligned}
\left(\xi_2^2\right)'' &= 2\,\xi_2\xi_2'' + 2\left(\xi_2'\right)^2 \\
&= 2\,\xi_2\xi_2'' + \frac{1}{2\xi_2^2}\left(2\,\xi_2\xi_2'\right)^2 \\
&= 2\,\xi_2\xi_2'' + \frac{1}{2\xi_2^2}\left[\left(\overline{x^2}\right)'\right]^2 \\
&= 2\,\xi_2\xi_2'' + \frac{2}{\xi_2^2}\left(\overline{xp}\right)^2
\end{aligned}$$

Therefore

$$\xi_2'' + \omega^2\,\xi_2 = \frac{\overline{x^2}\;\overline{p^2} - \left(\overline{xp}\right)^2}{\xi_2^3} \equiv \frac{E_2}{\xi_2^3} \tag{18}$$

Similar procedure gives

$$\xi_n'' + \omega^2 \xi_n = (n-1) \frac{E_n}{\xi_n^{2n-1}} \tag{19}$$

where $\xi_n \equiv \left(\overline{x^n}\right)^{\frac{1}{n}}$ and

$$\begin{cases} E_1 = 0 \\ E_2 = \overline{x^2}\ \overline{p^2} - \left(\overline{xp}\right)^2 \\ E_3 = \overline{x^3}\ \overline{xp^2} - \left(\overline{x^2 p}\right)^2 \\ E_4 = \overline{x^4}\ \overline{x^2 p^2} - \left(\overline{x^3 p}\right)^2 \\ E_5 = \overline{x^5}\ \overline{x^3 p^2} - \left(\overline{x^4 p}\right)^2 \end{cases} \tag{20}$$

Only the equation for ξ_2 is useful, because E_2 is identical to the invariant I_2. All other E_n obey rather complex time equations. The ξ_2 equation is reminiscent of the amplitude ($w \equiv \sqrt{\beta}$) equation of Courant and Snyder or the Kapchinsky-Vladimirsky equation in the absence of space charge. The ξ_2 equation may be called the rms envelope equation.

GENERALIZATIONS

Several straightforward generalizations should be pursued.

(a) When k is time dependent $k = k(t)$ we should write the single particle equation as

$$\begin{cases} x' = a(t)p, \\ p' = -b(t)x, \end{cases} \qquad K_1 \equiv \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}, \qquad \omega^2 \equiv ab \tag{21}$$

and proceed in a similar manner.

(b) When space-charge force is present we write

$$\begin{cases} x' = ap \\ p' = -bx + Fx \end{cases} \qquad K_1 \equiv \begin{pmatrix} 0 & a \\ -b+F & 0 \end{pmatrix} \qquad (22)$$

where $F = F(x,t)$ is the space-charge force. In this case a condition must be imposed on F to insure the invariance of I_2. This generalization can proceed in the manner a la Sacherer.

(c) The generalization to more than one coupled dimensions can be made in a straightforward way as indicated by Sacherer.